Note on blocks of p-solvable groups with same Brauer category

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Let p be a prime and let $\mathcal{O}$ be a complete discrete valuation ring with an algebraically closed residue field k of characteristic p. Let G be finite group and b be a block of G with maximal (G,b)-subpair (P,e_P) where b is a block idempotent of $\mathcal{O}G$. For any subgroup Q of P, let (Q,e_Q) be a unique (G,b)-subpair contained in (P,e_P). Following Kessar, Linckelmann and Robinson [4], we denote by $\mathcal{F}_{(P,e_P)}(G,b)$ the category whose objects are subgroups of P and for Q, $R \leq P$, whose set of morphisms from Q to R are the set of group homomorphisms $\varphi : Q \to R$ such that there exists $x \in G$ such that ${}^x(Q,e_Q) \subseteq (R,e_R)$ and $\varphi(u) = xux^{-1}$ for all $u \in Q$. We call $\mathcal{F}_{(P,e_P)}(G,b)$ the Brauer category of b. Let $\mathbf{B}_G(b)$ be the Brauer category of b in the sense of Thévenaz [10], § 47. The categories $\mathcal{F}_{(P,e_P)}(G,b)$ and $\mathbf{B}_G(b)$ are equivalent. Let R be a normal subgroup of P such that $N_G(P) \subseteq N_G(R)$ and c be the Brauer correspondent of b in $N_G(R)$, that is, c is a unique block of $N_G(R)$ such that $\mathrm{Br}_P(c) = \mathrm{Br}_P(b)$ where Br_P is the Brauer homomorphism from $(\mathcal{O}G)^P$ onto $kC_G(P)$. Set $N = N_G(R)$. The notations R, c and N are fixed. Thus $b = c^G$ and (P,e_P) is a maximal (N,c)-subpair. The arguments in the proof of Theorem in Kessar-Linckelmann [5] imply the following.

Theorem 1 *Assume that G is p-solvable. With the above notations, suppose that $\mathcal{F}_{(P,e_P)}(G,b) = \mathcal{F}_{(P,e_P)}(N,c)$. Then there is an indecomposable $\mathcal{O}Gb$-$\mathcal{O}Nc$-bimodule M which satisfies the following.*

(i) *M and its $\mathcal{O}$-dual M^* induce a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Nc$.*

(ii) *As an $\mathcal{O}(G \times N)$-module M has a vertex ΔP and an endo-permutation $\mathcal{O}(\Delta P)$- module as a source where $\Delta P = \{(u,u) \mid u \in P\}$.*

Let $H^*_{(P,e_P)}(G,b)$ be the cohomology ring of b in the sense of Linckelmann[6], [7], that is, $H^*_{(P,e_P)}(G,b)$ is the subring of $H^*(P,k)$ consisting of $\zeta \in H^*(P,k)$ satisfying $\mathrm{res}_Q\, \zeta = {}^g\mathrm{res}_Q\, \zeta$ for all $Q \leq P$ and, for all $g \in N_G(Q,e_Q)$. We prove the following.

Theorem 2 *Assume that G is p-solvable. With the above notations, if $H^*_{(P,e_P)}(G,b) = H^*_{(P,e_P)}(N,c)$, then $\mathcal{F}_{(P,e_P)}(G,b) = \mathcal{F}_{(P,e_P)}(N,c)$.*

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We prove Theorem 1 using the following.

Lemma 1 (Harris-Linckelmann [3], Lemma 4.2) *Assume that G is p- solvable. For any p-subgroup Q of G, we have $O_{p'}(N_G(Q)) = O_{p'}(G) \cap N_G(Q) = O_{p'}(G) \cap C_G(Q) = O_{p'}(C_G(Q))$.*

Proposition 1 (Harris-Linckelmann [2], Proposition 3.1 (iii)) *Let G be a p-solvable group and b be a block of G such that b covers a G-invariant block of $O_{p'}(G)$. Then b is of principal type, that is, for any p-subgroup Q of G, $\mathrm{Br}_Q(b)$ is a block of $kC_G(Q)$.*

Proposition 2 (Fong[1]; Puig[9]) *Let G be a p-solvable group and b be a block of G with defect group P. Then the following holds.*

(i) *There is a subgroup H of G and an H-invariant block e of $O_{p'}(H)$ such that $O_{p'}(G)P \subseteq H$ and $\mathcal{O}Gb \cong \mathrm{Ind}_H^G(\mathcal{O}He)$ as interior G-algebras.*

(ii) *P is a Sylow p-subgroup of H and P is a defect group of e as a block of H. Moreover let (P, e'_P) be a maximal (H,e)-subpair and let $e_P = \mathrm{Tr}_{C_H(P)}^{C_G(P)}(e'_P)$. Then (P, e_P) is a maximal (G,b)-subpair.*

Note that in the above proposition $\mathcal{F}_{(P,e_P)}(G,b) = \mathcal{F}_{(P,e'_P)}(H,e)$ since $\mathcal{O}Gb \cong \mathrm{Ind}_H^G(\mathcal{O}He)$ as interior G-algebras.

Proposition 3 ([5], Proposition 6) *With the notations in the above proposition, let R be a subgroup of P such that $N_G(P) \subseteq N_G(R)$. Denote by c the Brauer correspondent of b in $N_G(R)$, and by f the Brauer correspondent of e in $N_H(R)$. Then f is an $N_H(R)$-invariant block of $O_{p'}(N_H(R))$ and $\mathcal{O}N_G(R)c \cong \mathrm{Ind}_{N_H(R)}^{N_G(R)}(\mathcal{O}N_H(R)f)$ as interior $N_G(R)$-algebras.*

The following is shown in the proof of Theorem in [5].

Theorem 3 (Kessar-Linckelmann) *Let G be a p-solvable group and b be a block of G with defect group P. Let R be a subgroup of P such that $N_G(P) \subseteq N_G(R)$ and let c be the Brauer correspondent of b in N where we set $N = N_G(R)$. If b covers a G-invariant block of $O_{p'}(G)$ and if $G = O_{p'}(G)N$, then there is an indecomposable $\mathcal{O}Gb$-$\mathcal{O}Nc$-bimodule M which satisfies the following.*

(i) *M and its $\mathcal{O}$-dual M^* induce a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Nc$.*

(ii) *As an $\mathcal{O}(G \times N)$-module M has a vertex ΔP and an endo-permutation $\mathcal{O}(\Delta P)$- module as a source.*

Proof of Theorem 1. We prove by induction on $|G|$. Let H, e, e'_P and e_P be as in Proposition 2, and let f be as in Proposition 3. We may assume that e_P's in Theorem 1 and Proposition 2 are equal by replacing H, e, e'_P and f, by H^x, e^x, $(e'_P)^x$ and f^x respectively for some $x \in N_G(P)$ if necessary. By Proposition 2,

$$\mathcal{F}_{(P,e_P)}(G,b) = \mathcal{F}_{(P,e'_P)}(H,e).$$

By Proposition 3, (P, e'_P) is a maximal $(N_H(R), f)$-subpair and

$$\mathcal{F}_{(P,e_P)}(N, c) = \mathcal{F}_{(P,e'_P)}(N_H(R), f).$$

So by the assumption we have $\mathcal{F}_{(P,e'_P)}(H, e) = \mathcal{F}_{(P,e'_P)}(N_H(R), f)$. Since $\mathcal{O}Gb \cong \mathrm{Ind}_H^G(\mathcal{O}He)$ as interior G-algebras, the $\mathcal{O}Gb$-$\mathcal{O}He$-bimodule $b\mathcal{O}Ge = \mathcal{O}Ge$ and the $\mathcal{O}He$-$\mathcal{O}Gb$- bimodule $e\mathcal{O}G$ induce a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}He$. Similarly the $\mathcal{O}Nc$-$\mathcal{O}N_H(R)f$-bimodule $\mathcal{O}Nf$ and the $\mathcal{O}N_H(R)f$-$\mathcal{O}Nc$-bimodule $f\mathcal{O}N$ induce a Morita equivalence between $\mathcal{O}Nc$ and $\mathcal{O}N_H(R)f$. Suppose that $H < G$. By the induction hypothesis for H and e, there is an indecomposable $\mathcal{O}He$-$\mathcal{O}N_H(R)f$- bimodule M_0 such that M_0 and M_0^* induce a Morita equivalence between $\mathcal{O}He$ and $\mathcal{O}N_H(R)f$, and that M_0 as an $\mathcal{O}(H \times N_H(R))$-module has a vertex ΔP and an endo-permutation $\mathcal{O}(\Delta P)$-module as a source. Set $M = b\mathcal{O}G \otimes_{\mathcal{O}He} M_0 \otimes_{\mathcal{O}N_H(R)f} \mathcal{O}Nc \cong M_0^{G \times N}$. Then M satisfies (i) and (ii) in Theorem 1. Therefore we may assume that $H = G$. Then $b = e$.

Let $Y = O_{p',p}(G)$. Then b is a G-invariant block of Y because $Y/O_{p'}(G)$ is a p-group. Furthermore we have $Y = O_{p'}(G)(Y \cap P)$. Set $Q = P \cap Y$. Then Q is a defect group of b as a block of Y. Now since G is constrained, $C_Y(Q) = C_G(Q)$. Therefore we see that (Q, e_Q) is a maximal (Y, b)-subpair. By the Frattini argument and the assumption that $\mathcal{F}_{(P,e_P)}(G, b) = \mathcal{F}_{(P,e_P)}(N, c)$,

$$G = N_G(Q, e_Q)Y \subseteq N_N(Q)C_G(Q)Y \subseteq NY \subseteq NO_{p'}(G).$$

So we have $G = NO_{p'}(G)$. This and Theorem 3 complete the proof.

Proof of Theorem 2. We prove by induction on $|G|$. Let H, e, e'_P and e_P be as in Proposition 2, and let f be as in Proposition 3. We may assume that e_P's in Theorem 2 and Proposition 2 are equal as in the proof of Theorem 1. Since $\mathcal{F}_{(P,e_P)}(G, b) = \mathcal{F}_{(P,e'_P)}(H, e)$ and $\mathcal{F}_{(P,e_P)}(N, c) = \mathcal{F}_{(P,e'_P)}(N_H(R), f)$ we have

$$H^*_{(P,e_P)}(G, b) = H^*_{(P,e'_P)}(H, e),$$

$$H^*_{(P,e_P)}(N, c) = H^*_{(P,e'_P)}(N_H(R), f).$$

From the assumption, we have $H^*_{(P,e'_P)}(H, e) = H^*_{(P,e'_P)}(N_H(R), f)$. Suppose that $H < G$. Then by the induction hypothesis, $\mathcal{F}_{(P,e'_P)}(H, e) = \mathcal{F}_{(P,e'_P)}(N_H(R), f)$, and hence $\mathcal{F}_{(P,e_P)}(G, b) = \mathcal{F}_{(P,e_P)}(N, c)$. Therefore we may assume that $H = G$. Then b covers a G-invariant block of $O_{p'}(G)$ and P is a Sylow p-subgroup of G. Note that the element $b \in \mathcal{O}O_{p'}(G)$.

From Proposition 1, b is of principal type. On the other hand, by Lemma 1, $\mathrm{Br}_R(b)$ is an N-invariant block idempotent of $kO_{p'}(N)$ and c is a lifting of $\mathrm{Br}_R(b)$ to $\mathcal{O}N$. So by Proposition 1, c is also of principal type. So we may assume that b is a principal block. Therefore by a theorem of Mislin [8], we obtain $\mathcal{F}_{(P,e_P)}(G, b) = \mathcal{F}_{(P,e_P)}(N, c)$. This completes the proof.

References

[1] P. Fong, On the characters of p-solvable groups, Trans. Amer. Math. Soc. **98**(1961), 263-284.

[2] M.E. Harris and Linckelmann, Splendid derived equivalences for blocks of finite p-solvable groups, J. London Math. Soc. (2) **62**(2000), 85-96.

[3] M.E. Harris and Linckelmann, On the Glauberman and Watanabe correspondences for blocks of finite p-solvable groups, Trans. Amer. Math. Soc. **354**(2002), 3435-3453.

[4] R. Kessar, M. Linckelmann and G.R. Robinson, Local control in fusion systems of p-blocks of finite groups, J. Algebra **257**(2002), 393-413.

[5] R. Kessar and M. Linckelmann, On blocks of strongly p-solvable groups, D. Benson: Groups, Representations and Cohomology Preprint Archive.

[6] M. Linckelmann, Transfer in Hochschild cohomology of blocks of finite groups, Algebr. Represent. Theory **2** (1999), 107-135.

[7] M. Linckelmann, Varieties in block theory, J. Algebra **215**(1999), 460-480.

[8] G. Mislin, On group homomorphisms inducing mod p-cohomology isomorphism, Comment. Math. Helv. **65**(1990), 454-461.

[9] L. Puig, Local block theory in p-solvable groups, Proceedings of Symp. Pure Math. **37**(1980), 385-388.

[10] J. Thévenaz, "G-algebras and modular representation theory", Oxford Sci. Publ., Clarendon Press, Oxford, 1955.