Eigenvector matrices of Cartan matrices for finite groups

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1. Introduction

Let G be a finite group, F be an algebraically closed field of characteristic $p > 0$, and B be a block of the group algebra FG with defect group D. Let $C_B = (c_{ij})$ be the Cartan matrix of B and $\rho(B)$ be the Frobenius-Perron eigenvalue (i.e. the largest eigenvalue) of C_B. Let (K, R, F) be a p-modular system, where R is a complete discrete valuation ring of rank one with $R/(\pi) \simeq F$ for a unique maximal ideal (π) and K is a quotient field of R with characteristic 0. Let us denote the number $l(B)$ of irreducible Brauer characters in B simply by l.

We studied on integrality of eigenvalues of the Cartan matrix of a finite group in [4], [17]. Let R_B and E_B be the set of all eigenvalues and $\mathbf{Z}$-elementary divisors of C_B, respectively. For cyclic blocks or tame blocks, we proved that $\rho(B) \in \mathbf{Z}$ if and only if $R_B = E_B$, and for any p-blocks of p-solvable groups, we proved that $\rho(B) = |D|$ if and only if $R_B = E_B$. Recently, C.C. Xi and D. Xiang proved that a cellular algebra A is semisimple if and only if all eigenvalues of the Cartan matrix of A are rational integers and the Cartan determinant is 1 ([15, Theorem 1.1]).

Then, what do eigenvectors of C_B mean? In this article the author showed that if all eigenvalues of C_B are rational integers for a cyclic block B, a tame block B, a p-block B of a p-solvable group or the principal 3-block B with elementary abelian Sylow p-subgroup of order 9, then there exists a unimodular matrix U_B over R whose columns consist of eigenvectors of C_B. We call U_B *an eigenvector matrix* of C_B. From Linear Algebra U_B diagonalizes C_B. In these cases above, we can take as U_B actually the Brauer character table matrix for some blocks. For details see [19].

2. Preliminaries

We had the following basic conjecture in [4].

Conjecture (Questions 1 and 2 in [4]). *Let G be a finite group. Let B be a block of FG with defect group D. Then the following are equivalent.*
(a) $\rho(B) \in \mathbf{Z}$.
(b) $\rho(B) = |D|$.
(c) $R_B = E_B$.

For several groups or blocks we proved that Conjecture is true, but we do not yet prove for any finite groups. If this conjecture is true, these conditions must be equivalent to

(d) *all eigenvalues of C_B are rational integers.*

Because, (d) implies (a) , and (c) implies (d). Here we try to consider proving $(d) \to (c)$ in the following section. To begin with, we state some preliminary results in [4]. We first introduce some notation. Let $\mathrm{IBr}(B) = \{\varphi_1, \cdots, \varphi_l\}$ be the set of irreducible Brauer characters in a block B of FG. Let $\{x_1, \cdots, x_l\}$ be a set of representatives of p-regular classes of G associated with B ([10, Theorem 11.6]). Let us set $\boldsymbol{\varphi}_j = {}^t(\varphi_1(x_j), \cdots, \varphi_l(x_j))$ for $1 \le j \le l$ and let $\Phi_B = (\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_l) = (\varphi_i(x_j))$ be *the Brauer character table* of B. Here for a matrix A we denote by ${}^tA$ the transposed matrix of A.

Theorem 1 (Proposition 2 in [4], see also [1, Lemma 4.26 (Lusztig)]). *Let B be a block of FG with defect group D. Suppose $D \lhd G$. Then the following hold.*
(1) $C_B\boldsymbol{f} = |D|\boldsymbol{f}$, *where* $\boldsymbol{f} = {}^t(\varphi_1(1), \cdots, \varphi_l(1))$ *for* $\{\varphi_1, \cdots, \varphi_l\} = \mathrm{IBr}(B)$.
(2) $R_B = E_B = \{|C_D(x_1), \cdots, |C_D(x_l)|\}$, *where* $\{x_1, \cdots, x_l\}$ *is a set of representatives of p-regular classes of G associated with B. In particular,*

$$C_B\Phi_B = \Phi_B \ \mathrm{diag}\{|C_D(x_1)|, \cdots, |C_D(x_l)|\}.$$

Theorem 2 (Theorem 1 in [4]). *Let G be a p-solvable group and let B be a block of FG with defect group D. Then the following are equivalent.*
(a) $\rho(B) = |D|$.

(*b*) $R_B = E_B$.
(*c*) *the height of* $\varphi = 0$ *for any* $\varphi \in \mathrm{IBr}(B)$.
(*d*) $\boldsymbol{f} = {}^t(\varphi_1(1), \cdots, \varphi_l(1))$ *is an eigenvector for* $\rho(B)$.

Theorem 3 (Proposition 3 in [4]). *Let B be a block of FG with a cyclic defect group D. Then the following are equivalent.*
(*a*) $\rho(B) \in \mathbf{Z}$.
(*b*) $\rho(B) = |D|$.
(*c*) $R_B = E_B$.
(*d*) *The Brauer tree* Γ_B *of B is a star and its exceptional vertex with multiplicity m, if it exists, is at the center. In this case*

$$C_B = \begin{pmatrix} m+1 & m & \cdots & m \\ m & m+1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & m \\ m & \cdots & m & m+1 \end{pmatrix}.$$

(*e*) *B is Morita equivalent to its Brauer correspondent block b of* $FN_G(D)$.

Theorem 4 (Proposition 4 in [4]). *Let B be a tame block (not finite type) of FG with defect group D (i.e.* $p = 2$ *and D is isomorphic to a dihedral, a generalized quaternion or a semidihedral group). Then the following are equivalent.*
(*a*) $\rho(B) \in \mathbf{Z}$.
(*b*) $\rho(B) = |D|$.
(*c*) $R_B = E_B$.
(*d*) *One of the following holds.*
  (i) $l = 1$,
  (ii) $l = 3, D \simeq E_4$ *(an elementary abelian group of order four) and*

$$C_B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

  (iii) $l = 3, D \simeq Q_8$ *(a quaternion group of order eight) and*

$$C_B = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

(*e*) *B is Morita equivalent to its Brauer correspondent block b of* $FN_G(D)$.

3. All eigenvalues of C_B are integers

Lemma 1 ([5, Proposition 4.5]). *Let B be a block of FG with defect group D. Let λ be an eigenvalue of C_B. Then there is an algebraic integer μ such that $\lambda\mu = |D|$. In particular, if λ is a rational integer, then λ is a power of p dividing $|D|$.*

From Linear Algebra there exists a non singular matrix U_B over the field $\mathbf{R}$ of real numbers whose column vectors consist of linearly independent l eigenvectors of C_B such that $U_B^{-1}C_BU_B = \mathrm{diag}\{\rho_1, \cdots, \rho_l\}$ since C_B is a real symmetric matrix. We assume that *all eigenvalues $\rho_1, \cdots, \rho_l$ of C_B are rational integers*. Then ρ_i is a power of p for $1 \leq i \leq l$ by Lemma 1. We note that in this case we can have an eigenvector $\boldsymbol{u}_i$ of ρ_i being in $\mathbf{Z}^l$. Suppose further that $U_B = (\boldsymbol{u}_1, \cdots, \boldsymbol{u}_l)$ *can be taken as a unimodular matrix over the complete discrete valuation ring R (i.e. $U_B \in \mathrm{GL}(l, R)$)*. Then since $\rho_1, \cdots, \rho_l$ are powers of p, they are also $\mathbf{Z}$-elementary divisors of C_B because $U_B^{-1}C_BU_B = \mathrm{diag}\{\rho_1, \cdots, \rho_l\}$ and U_B is unimodular. Thus $R_B = E_B$. So the following question naturally arises.

Question 1. Let G be a finite group and B be a block of FG. Let C_B be the Catan matrix of B. Then can we take a unimodular eigenvector matrix U_B of C_B over R?

At least does the following hold?

Question 2. Furthermore suppose that all eigenvalues $\rho_1, \cdots, \rho_l$ of C_B are rational integers. Then can we take a unimodular eigenvector matrix U_B of C_B over R ? i.e. Does there exist $U_B \in \mathrm{Mat}_l(\mathbf{Z})$ such that $\det U_B \not\equiv 0 (\mathrm{mod} p)$?

We note that there exists a negative example for Question 2 in a general finite dimensional algebra which is not a finite group algebra.

Example ([16]). Let B be a Brauer tree algebra. Let Γ_B be the Brauer tree $\bullet - \circ - \bullet$ with three vertices, where $\bullet$ means an exceptional vertex with multplicity m. Then we have

$C_B = \begin{pmatrix} m+1 & 1 \\ 1 & m+1 \end{pmatrix}$. So $R_B = \{m+2, m\}$, $\quad E_B = \{m^2+2m, 1\}$.

Thus eigenvalues of C_B are rational integers, but $R_B \neq E_B$ if $m > 1$. Actually, we can take an eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $m+2$ and an eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ for m. So we can take a eigenvector matrix $U_B = \begin{pmatrix} \alpha & -\beta \\ \alpha & \beta \end{pmatrix}$ over R of C_B for $\alpha, \beta \in R$, then $\det U_B = 2\alpha\beta$. Therefore, if $p = 2$, $\det U_B \equiv 0 \pmod{(\pi)}$. Thus, if $p = 2$, we can never take a unimodular eigenvector matrix of C_B over R.

If the above C_B appears as the Cartan matrix of a 2-block of a finite group, $\det C_B$ must be a power of 2. So $m = 2$ and $C_B = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. However, the following results show that this matrix cannot be the Cartan matrix for cyclic blocks, tame blocks, p-blocks of p-solvable groups, at least.

4. Theorems

We have the following results on Question 2.

Theorem A. *Let B be a cyclic block or a tame block. If $\rho(B) \in \mathbf{Z}$, then we can take a unimodular eigenvector matrix U_B of C_B over R. Indeed we have $U_B = \Phi_b$, where b is the Brauer correspondent block of B.*

Theorem B. *Let G be a p-solvable group. If $\rho(B) = |D|$, then we can take a unimodular eigenvector matrix U_B of C_B over R. Indeed we have $U_B = \Phi_\beta$ for some block β of a subgroup of G or a factor group of a central extension of G.*

Proof of Theorem A. Let B be a cyclic block or a tame block of G. Then by Theorems 3 and 4 we have that B and its Brauer correspondent block b are Morita equivalent. Thus $C_B = C_b$. Then we can take $U_B = \Phi_b$ by Theorem 1, which is unimodular over R. $\square$

We use the Fong reduction to prove Theorem B, but we omit it. The following result is due to Koshitani-Kunugi [7] and many author's results (e.g. [8,9,12,13]) through proving Broué's abelian defect group conjecture to be true.

Theorem C. *Let $\widetilde{G}$ be a finite group with an elementary abelian Sylow 3-subgroup P of order 9. Let $\widetilde{B}$ and $\widetilde{b}$ be the principal 3-block of $\widetilde{G}$ and $N_{\widetilde{G}}(P)$, respectively. Suppose $O_{3'}(\widetilde{G}) = 1$. Then the following are equivalent.*
(*a*) $\rho(\widetilde{B}) \in \mathbf{Z}$.
(*b*) $\rho(\widetilde{B}) = |P| = 9$.
(*c*) $R_{\widetilde{B}} = E_{\widetilde{B}}$.
(*d*) $\widetilde{B}$ *and* $\widetilde{b}$ *are Morita equivalent (even stronger Puig equivalent).*
(*e*) *Let* $G := O^{3'}(\widetilde{G})$. *Then one of* (i) *and* (ii) *holds.*
(i) $G = X \times Y$, *where* X, Y *are simple groups (abelian or not) with a cyclic Sylow 3-subgroup of order 3, respectively.*
(ii) G *is one of the following non abelian simple groups with elementary abelian Sylow 3-subgroup of order 9.*
(1) $\mathrm{PSU}_3(q^2)$ *with* $2 < q \equiv 2$ *or* $5 \pmod 9$.
(2) $\mathrm{PSp}_4(q)$ *with* $q \equiv 4$ *or* $7 \pmod 9$.
(3) $\mathrm{PSL}_5(q)$ *with* $q \equiv 2$ *or* $5 \pmod 9$.
(4) $\mathrm{PSU}_4(q^2)$ *with* $q \equiv 4$ *or* $7 \pmod 9$.
(5) $\mathrm{PSU}_5(q^2)$ *with* $q \equiv 4$ *or* $7 \pmod 9$.
In these cases, we can take $\Phi_{\widetilde{b}}$ *as a unimodular eigenvector matrix* $U_{\widetilde{B}}$ *of* $C_{\widetilde{B}}$.

Proof. (e) $\rightarrow$ (d). Then [7, (5.3),(5.6)] states that $\widetilde{B}$ and $\widetilde{b}$ are Puig equivalent.

It is easy to see that (d) $\rightarrow$ (c), because $C_B = C_b$ and by Theorem 1. It is obvious that (c) $\rightarrow$ (b) and (b) $\rightarrow$ (a).

(a) $\rightarrow$ (e). Suppose (e) does not hold. Then G is one of the following alternating groups or sporadic simples $A_6, A_7, A_8, M_{11}, M_{22}, M_{23}, M_{24}, HS$ or one of the following simple groups of Lie type ; $\mathrm{PSL}_3(q)$ for $q \equiv 4$ or $7 \pmod 9$, $\mathrm{PSp}_4(q)$ for $2 < q \equiv 2$ or $5 \pmod 9$, $\mathrm{PSL}_4(q)$ for $2 < q \equiv 2$ or $5 \pmod 9$. But for these simple groups we can easily check that $\rho(B_0(FG))$ is not a rational integer, here B_0 means the principal 3-block. So we can prove that neither is $\rho(B_0(F\widetilde{G}))$ by the following Proposition.

Proposition 3. *Let G be a finite group and $H \lhd G$ with $|G : H| = q$, where q is a prime number distinct from p. Let b be a block of FH and $B_1, \cdots, B_m$ be all blocks of G covering b. Then $\rho(B_i) = \rho(b)$ for all $1 \leq i \leq m$.*

We skip to prove Proposition 3. Here for example we show a typical case. Other cases are similar.

If $G = M_{11}$, then $C = \begin{pmatrix} 5 & 2 & 2 & 3 & 3 & 0 & 1 \\ 2 & 3 & 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & 3 & 1 & 2 & 0 & 1 \\ 3 & 2 & 1 & 4 & 3 & 1 & 2 \\ 3 & 1 & 2 & 3 & 4 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 2 & 1 & 2 \end{pmatrix}$ and

$$f_C(x) = (x^2 - 3x + 1)(x^5 - 20x^4 + 102x^3 - 192x^2 + 135x - 27).$$

Since $10 < \rho(B) < 16$, $\rho(B)$ is not an integer by [5, Lemma 3.1 (2)].

Thus we have proved (a) $\rightarrow$ (e). Then by Lemma 2 we can take $U_B = \Phi_b$. So there exists a unimodular eigenvector matrix of C_B in this case. $\square$

Remark 1. There are small misprints in [7, (5.7) Lemma]. In Case 2 and Case 4 the small star marks should be the big star marks.

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