Type transformations for sharp characters

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1 Introduction

Let G be a finite group and χ be a faithful character of G of degree n. Put $L = \{\chi(g) \mid g \in G,\, g \neq 1\}$. Then we have the following

Theorem 1 (Blichfeldt[B]) $|G|$ divides the integer $\prod_{l \in L}(n-l)$.

Theorem 1 gives us the upper bound of the order of G. We are interested in the case G attains the bound.

Definition 1 We call $(G,\, \chi)$ sharp of type L (or L-sharp) if $|G| = \prod_{l \in L}(n-l)$ holds.

Problem 1 For a given L, determine all L-sharp pairs $(G,\, \chi)$.

Example 1 Let G be a sharply t-transitive permutation group, which is different from S_t, the symmetric group of degree t. Let π be the permutation character of G. Then $(G,\, \pi)$ is sharp of type $\{0, 1, \cdots, t-1\}$.

Note that $(G,\, \chi)$ is sharp if and only if $(G,\, \chi + 1_G)$ is sharp, where 1_G is the trivial character of G. So we may assume $(\chi, 1_G) = 0$ holds, when we consider sharp characters χ. We call such character normalized sharp character.

We have the following results concerning Problem 1. When L contains an irrational number, L-sharp pairs $(G,\, \chi)$ are completely classified by Alvis-Nozawa[A-N]. Hence we may assume that $L \subset \mathbf{Z}$ holds. The cases $L = \{l\},\ \{l, l+1\},\ \{l, l+2\},\ \{l, l+1, l+2\},\ \{l, l+1, l+2, l+3\}$ are treated in Cameron-Kiyota [C-K]、Cameron-Kataoka-Kiyota [C-K-K]、Nozawa [N]. We do not have any classification results for "big" L in case $L \subset \mathbf{Z}$, and so we should ask the following

Problem 2 Can we reduce the classification of L-sharp pairs to that of L'-sharp pairs for some L' with $|L'| < |L|$?

2 Transformations of types

Let $L_1,\ L_2$ be finite sets of complex numbers with $|L_1| = |L_2| = m \geq 2$.

Definition 2 We write $L_1 \sim L_2$ if $e_1(L_1) = e_1(L_2),\ e_2(L_1) = e_2(L_2), \cdots, e_{m-1}(L_1) = e_{m-1}(L_2)$ hold, where $e_k(L_1)$ is the k-th elementary symmetric function with variables in L_1. For example, $e_1(L_1) = \sum_{l \in L_1} l,\ e_m(L_1) = \prod_{l \in L_1} l$.

Example 2 $\{a, b\} \sim \{c, d\} \Longleftrightarrow a + b = c + d$,

$\{a, b, c\} \sim \{d, e, f\} \Longleftrightarrow a + b + c = d + e + f,\ ab + bc + ca = de + ef + fd$

The following two lemmas are fundamental but easy to prove.

Lemma 1
(1) $L_1 \sim L_2 \Longleftrightarrow L_1 + l \sim L_2 + l$, where we denote $L_1 + l = \{a + l \,|\, a \in L_1\}$.
(2) If $L_1 \sim L_2$, then we have

$$L_1 = L_2 \Longleftrightarrow L_1 \cap L_2 \neq \emptyset \Longleftrightarrow e_m(L_1) = e_m(L_2).$$

Lemma 2 Assume $L \subset \mathbf{C},\ \ |L| = rm\ (m \geq 2)$. Then the followings are equivalent.
(1) There exists a monic polynomial $f(X) \in \mathbf{C}\,[X]$ of degree m with $|f(L)| = r$.
(2) There exists a decomposition of L, $L = L_1 \cup \cdots \cup L_r$ with $|L_k| = m,\ L_1 \sim \cdots \sim L_r$.

Using the above lemmas, we can prove the following Theorem.

Theorem 2 Let χ be a faithful character of a finite group G. Set $L = \{\chi(g) \,|\, g \in G,\ g \neq 1\}$. Suppose that there exists a decomposition of L, $L = L_1 \cup \cdots \cup L_r$ with $|L_k| = m \geq 2,\ L_1 \sim \cdots \sim L_r$. Assume further that each L_k is algebraically closed. Then there exists a monic $f(X) \in \mathbf{Z}\,[X]$ which satisfies the following two conditions.

(i) $(G,\ \chi)$ is sharp of type $L \quad \Longleftrightarrow \quad (G,\ f(\chi))$ is sharp of type $f(L)$.

(ii) $f(L) = \{(-1)^{m-1} e_m(L_1), \cdots, (-1)^{m-1} e_m(L_r)\}$.

We will give some examples that shows how to apply Theorem 2.

Example 3 Let (G, χ) be normalized sharp of type $L = \{-1, 0, 1, 2\}$. Note that $L = \{-1, 2\} \cup \{0, 1\}$, $\{-1, 2\} \sim \{0, 1\}$. So L satisfies the conditions of Theorem 2. If we put $f(X) = X^2 - X$, then $(G, f(\chi))$ is sharp of type $\{2, 0\}$ (but not necessarily normalized). Using the classification of sharp of type $\{l, l+2\}$, we get $G = S_5, A_6, M_{11}$, Thus, G is a sharply 4-transitive group except S_4.

Example 4 $L = \{-1, 0, 2, 3\} = \{-1, 3\} \cup \{0, 2\}$ satisfies the conditions of Theorem 2. Using $f(X) = X^2 - 2X$, we can reduce the determination of L-sharp pairs to that of $\{3, 0\}$-sharp pairs. But unfortunately we do not have complete classification of $\{l, l+3\}$-sharp pairs.

Example 5 $L = \{-2, -1, 0, 2, 3, 4\} = \{-1, 0, 4\} \cup \{-2, 2, 3\}$ satisfies the conditions of Theorem 2. Using $f(X) = X^3 - 3X^2 - 4X$, we can reduce the determination of L-sharp pairs to that of $\{0, -12\}$-sharp pairs. But again we do not have complete classification of $\{l, l+12\}$-sharp pairs.

Remarks In Theorem 2, $f(\chi)$ is a generalized character of G and is not necessarily character. $f(\chi)$ is not necessarily normalized, even if χ is so.

References

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