On a class of rigid Coxeter groups

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The purpose of this note is to introduce some results of recent papers [4] and [5] about rigid Coxeter groups.

A *Coxeter group* is a group W having a presentation

$$\langle\, S \,|\, (st)^{m(s,t)} = 1 \text{ for } s,t \in S \,\rangle,$$

where S is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

(i) $m(s,t) = m(t,s)$ for any $s,t \in S$,
(ii) $m(s,s) = 1$ for any $s \in S$, and
(iii) $m(s,t) \geq 2$ for any $s,t \in S$ such that $s \neq t$.

The pair (W,S) is called a *Coxeter system*. For a Coxeter group W, a generating set S' of W is called a *Coxeter generating set for* W if (W,S') is a Coxeter system. Let (W,S) be a Coxeter system. For a subset $T \subset S$, W_T is defined as the subgroup of W generated by T, and called a *parabolic subgroup*. A subset $T \subset S$ is called a *spherical subset of* S, if the parabolic subgroup W_T is finite.

Let (W,S) and (W',S') be Coxeter systems. Two Coxeter systems (W,S) and (W',S') are said to be *isomorphic*, if there exists a bijection $\psi : S \to S'$ such that

$$m(s,t) = m'(\psi(s), \psi(t))$$

for every $s,t \in S$, where $m(s,t)$ and $m'(s',t')$ are the orders of st in W and $s't'$ in W', respectively.

A *diagram* is an undirected graph Γ without loops or multiple edges with a map $\mathrm{Edges}(\Gamma) \to \{2, 3, 4, \dots\}$ which assigns an integer greater than 1 to each of its edges. Since such diagrams are used to define Coxeter systems, they are called *Coxeter diagrams*.

In general, a Coxeter group does not always determine its Coxeter system up to isomorphism. Indeed some counter-examples are known.

Example ([1, p.38 Exercise 8], [2])**.** It is known that for an odd number $k \geq 3$, the Coxeter groups defined by the diagrams in Figure 1 are isomorphic and D_{2k}.

FIGURE 1. Two distinct Coxeter diagrams for D_{2k}

Example ([2])**.** It is known that the Coxeter groups defined by the diagrams in Figure 2 are isomorphic by the *diagram twisting* ([2, Definition 4.4]).

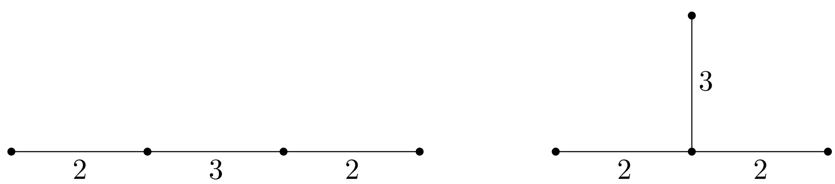

FIGURE 2. Coxeter diagrams for isomorphic Coxeter groups

Here there exists the following natural problem.

Problem ([2], [3])**.** When does a Coxeter group determine its Coxeter system up to isomorphism?

A Coxeter group W is said to be *rigid*, if the Coxeter group W determines its Coxeter system up to isomorphism (i.e., for each Coxeter generating sets S and S' for W the Coxeter systems (W, S) and (W, S') are isomorphic).

A Coxeter system (W, S) is said to be *even*, if $m(s, t)$ is even for all $s \neq t$ in S. Also a Coxeter system (W, S) is said to be *strong even*, if $m(s, t) \in \{2\} \cup 4\mathbb{N}$ for all $s \neq t$ in S.

The following theorem was proved by Radcliffe in [6].

Theorem 1 ([6])**.** *If (W, S) is a strong even Coxeter system, then the Coxeter group W is rigid.*

In [4], we first proved the following theorem which give a new class of rigid Coxeter groups.

Theorem 2. *Let (W, S) be a Coxeter system. Suppose that*

(0) *for each $s, t \in S$ such that $m(s, t)$ is even, $m(s, t) = 2$,*
(1) *for each $s \neq t \in S$ such that $m(s, t)$ is odd, $\{s, t\}$ is a maximal spherical subset of S,*
(2) *there does not exist a three-points subset $\{s, t, u\} \subset S$ such that $m(s, t)$ and $m(t, u)$ are odd, and*
(3) *for each $s \neq t \in S$ such that $m(s, t)$ is odd, the number of maximal spherical subsets of S intersecting with $\{s, t\}$ is at most two.*

Then the Coxeter group W is rigid.

Example. The Coxeter groups defined by the diagrams in Figure 3 are rigid by Theorem 2.

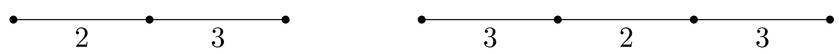

FIGURE 3. Coxeter diagrams for rigid Coxeter groups

In [5], we also proved the following theorem which is an extension of Theorem 1 and Theorem 2.

Theorem 3. *Let* (W, S) *be a Coxeter system. Suppose that*

(0) *for each* $s, t \in S$ *such that* $m(s,t)$ *is even,* $m(s,t) \in \{2\} \cup 4\mathbb{N}$*,*

(1) *for each* $s \neq t \in S$ *such that* $m(s,t)$ *is odd,* $\{s,t\}$ *is a maximal spherical subset of* S*,*

(2) *there does not exist a three-points subset* $\{s,t,u\} \subset S$ *such that* $m(s,t)$ *and* $m(t,u)$ *are odd, and*

(3) *for each* $s \neq t \in S$ *such that* $m(s,t)$ *is odd, the number of maximal spherical subsets of* S *intersecting with* $\{s,t\}$ *is at most two.*

Then the Coxeter group W *is rigid.*

References

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